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Cohomological results in monoid and category theory via classifying spaces

Marc Nunes

Department of Mathematics, University of California, Berkeley, CA 94720, USA

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Abstract

The thesis lays foundations for a topological, homotopical, and homological methodology in the study of monoids via the classifying space functor $|\ |: \mathrm{CAT} \to \mathrm{TOP}$.

In Section 0 basic constructions are described giving relationships between categories, monoid actions, and posets. An important adjunction $D: \mathrm{CAT}\downarrow C \rightleftarrows \mathrm{CAT}\, C^{\mathrm{op}}: *$ is described that will be used extensively throughout the work. Then the classifying space functor is introduced and several known homotopy theoretical results are stated. Using these results, it is shown that when the classifying space functor is applied to the adjunction above, one gets maps that are homotopy equivalences. This has far-reaching implications. The (co)homology of the classifying spaces is then described and it is proved that this (co)homology arises from the comonad associated with the above adjunction. This is a direct generalization of the fact that the bar construction arises from an adjunction. Let X be a right M-set with M a given monoid. Then it is proved that $H_*(M, ZX)$ is isomormphic to the classifying space of a certain category naturally associated with the (M, X). The section ends with conditions under which the classifying spaces of a monoid is homotopy equivalent to a finite-dimensional CW-complex.

In Section 1, it is proved that associated to every functor $F: X \to Y$ is a spectral sequence whose $E^2_{p,q}$ term is $H_p(Y^{\mathrm{op}}, H_q DF)$ and whose termination is $H_{p+q}(X, Z)$. Here, $DF: Y^{\mathrm{op}} \to \mathrm{CAT}$ is the functor associated with F where D is as in the above adjunction. Thus, $H_q DF = H_q(DF(\ ), Z) \to AB$ as a functor. One should anticipate this result, since $|X| \simeq |DF * Y|$ from Section 0, and $DF * Y \to Y$ should be thought of as a categorical analogue of a fiber bundle with "fiber" DF. Then, using the results of Section 0, we obtain that given $f: C^{\mathrm{op}} \to \mathrm{CAT}$, there exists a spectral sequence whose $E^2_{p,q}$ term is $H_q(C^{\mathrm{op}}, H_q f)$ and whose termination is $H_{p+q}(|f * C|, \mathbf{Z})$. Here, $H_q f: C^{\mathrm{op}} \to Ab$ is given by $H_q f(\ ) = H_q(|f(\cdot)|, \mathbf{Z})$. The first spectral sequence is applied to the following situation to compute the Euler characteristic of $|C|$ when it exists: $F: C \to P$ is a functor from C to a finite poset such that $H_q DF = 0$ if q is large enough, and finitely generated otherwise. The section ends with conditions under which the right M-module $\mathbf{Z}$ where $i \cdot m = 0$ if $m \neq 1$ (we assume here that the group of units of $M = \{1\}$) is acyclic. This module is of importance in the next section.

The last section deals with the issue of when a surmorphism is a (co)homological equivalence. It is shown that $f: M \to N$ is a homology equivalence iff $|Df|$ is acyclic. Then, it is proved that if $f: M \to N$ is given and $I \subseteq M$ is an ideal such that $f|M - I$ is injective as a function, then if $f|I$ is a homology equivalence, so if f. This result, proved using the spectral sequence of Section 1, should be compared to the following result in homotopy theory: Let (X, A) be an NDR pair and suppose $f: A \to B$ is a homotopy equivalence. Then so is $\hat{f}: X \to X \cup_f B$. Then *null*

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surmorphisms are defined and it is proved that these surmorphisms are (co)homology equivalences.

0.1. Basic constructions in category theory and monoid actions

Recall that if C and D are categories, then D^C is the category whose objects are functors $f: C \to D$ and whose arrows are given by natural transformations $\eta: f_1 \to f_2$. If $f: C_1 \to C_2$ is a functor, then we have an induced functor $D^F: D^{C_2} \to D^{C_1}$ given by $D^F(f) = fF: C_1 \to D$ and if $\eta: f_1 \to f_2$ is an arrow in D^{C_2} then $D^F(\eta) = \eta F: f_1 F \to f_2 F$.
On the other hand, if $G: D_1 \to D_2$ is a functor, then we have a induced functor $G^C: D_1^C \to D_2^C$ given by $G^C(f) = Gf: C \to D_2$ and if $\eta: f_1 \to f_2$ is an arrow in D_1^C, then $G^C(\eta) = G\eta: Gf_1 \to Gf_2$.

Now suppose C is a monoid. Then the objects of D^C are representations $f: C \to \mathrm{Hom}(d, d)$ for some $d \in \mathrm{Obj}(D)$. The arrows are then given as follows: if $f_1: C \to \mathrm{Hom}(d_1, d_1)$ and $f_2: D \to \mathrm{Hom}(d_2, d_2)$ are two representations, then an arrow $a: d_1 \to d_2$ such that $af_1 = f_2 a$ is an arrow in D^C.
In the following M will denote a monoid.

Example 0.1.1. If $D = AB$, then objects in AB^M are right M-modules and the arrows are M-linear homomorphisms.

Example 0.1.2. If $D = \mathrm{SET}$, then the objects of D^M are right M-sets and the arrows are equivariant maps.

Example 0.1.3. If $D = \mathrm{Htpy}$ the homotopy category is one whose objects are topological spaces and whose arrows are given by homotopy classes of maps between spaces. Then an object in Htpy^M is given by a space X and a map $f: M \to C^0(X, X)$ such that $f(m)f(n)$ is homotopic to $f(mn)$. An arrow is given by a continuous map $g: X \to Y$ such that $g(xm)$ is homotopic to $g(x)m$.

Now let $D = \mathrm{SET}$. Then for an object in SET^M, i.e. a right M-set X, we have a naturally associated set of orbits $\Omega_M(X) = \{O(x)\}$ where $O(x) = \{xm \mid m \in M\}$. In the case where M is a group, $\Omega_M(X)$ is a partition of X. On the other hand, if M is a general monoid, $\Omega_M(X)$ will in general be a *cover* of X, since orbits can intersect nontrivially.
Suppose $a: f_1 \to f_2$ is an arrow in SET^M. This is an equivariant map $a: X \to Y$. Define $\Omega_M(a): \Omega_M(Y)$ by $\Omega_M(a)(O(x)) = O(a(x))$. Then if $O(x_1) \subseteq O(x_2)$, $x_1 = x_2 m$ for some m in M. Then $a(x_1) = a(x_2 m) = a(x_2)m$ and so $O(a(x_1)) \subseteq O(a(x_2))$. It is trivial to see that if a_1 and a_2 are composable arrows, then $\Omega_M(a_1 a_2) = \Omega_M(a_1)\Omega_M(a_2)$.

We thus have the following proposition.

Proposition 0.1.4. $\Omega_M : \mathrm{SET}^M \to \mathrm{Poset}$ *is a functor.*

There is another way of viewing this that will be conceptually useful in what follows. Let X be a right M-set. Define a preorder on X by saying $x < y$ iff there exists $m \in M$ so that $x = ym$. It is easy to see that the poset associated with this preorder is isomorphic to $\Omega_M(X)$.

A similar construction applies to CAT. Namely, if C is a category, define a preorder on $\mathrm{Obj}(C)$ by saying $c_1 < c_2$ iff there exists an arrow $a : c_2 \to c_1$. We then pass to the associated poset and call it $\Omega(C)$. This clearly gives a functor $\Omega : \mathrm{CAT} \to \mathrm{Poset}$.

On the other hand, given a right M-set X, we have a naturally associated category $X//M$ whose objects are given by the set X and whose arrows are given by $m : x_1 \to x_2$ if $x_2 = x_1 m$. It is a simple matter to check that this assignment is functorial in X. We denote this functor by $//M : \mathrm{SET}^M \to \mathrm{CAT}$.

It is also easy to see that we have the following commutative diagram

$$\begin{tikzcd} & \mathrm{CAT} \arrow[d, "\Omega"] \\ \mathrm{SET}^M \arrow[ur, "M"] \arrow[r, "\Omega_M"'] & \mathrm{Poset} \end{tikzcd}$$

Example 0.1.5. Let M act on M via the right regular representation. Then $\Omega_M(M) = R_M$, the R-order poset induced by the R Green's relation. Likewise, if M acts by the left regular representation on M, then $\Omega_{M^{\mathrm{op}}}(M) = L_M$, the L-order poset induced by the L Green's relation. Finally, let $M^{\mathrm{op}} \times M$ act on M by $(m, n)n' = mn'n$. Then $\Omega_{M^{\mathrm{op}} \times M}(M) = J_M$, the J-order induced by the J Green's relation.

0.2. The Grothendieck–Kan–Nico–Tilson adjunction

This is an adjunction associated with a given category C of the following form:

$$\begin{tikzcd} \mathrm{CAT}\downarrow C \arrow[r, shift left, "D"] & \mathrm{CAT}^{C^{\mathrm{op}}} \arrow[l, shift left, "*"] \end{tikzcd}.$$

It is called the above because the cited mathematicians have discovered it independently. I shall describe it but will leave out many details not central to what follows. The interested reader may consult [7, 11]. $D : \mathrm{CAT}\downarrow C \to \mathrm{CAT}^{C^{\mathrm{op}}}$ is given as follows: If $F : D \to C$ is an object in $\mathrm{CAT}\downarrow C$, then $DF : C^{\mathrm{op}} \to \mathrm{CAT}$ is defined by $DF(c) = c\downarrow F$ on objects and if $a' : c_1 \to c_2$, then $DF(a') : DF(c_2) \to DF(c_1)$ is given by

$$DF(a')\,(a', d) = (a'a, d)$$

an objects and

$$DF(a')(b:(a_1, d_1) \to (a_2, d_2)) = b:(a'a_1, d_1) \to (a'a_2, d_2)$$

on arrows.

Next, $*: \mathrm{CAT}^{C^{\mathrm{op}}} \to \mathrm{CAT}\downarrow C$ is defined on objects as follows: If $f: C \to \mathrm{CAT}$, then $\mathrm{Obj}(f * C) = \{(c, x) | x \in \mathrm{Obj}(f(c))\}$. The arrows of $f * C$ are

$$(a, u):(c_1, x_1) \to (c_2, x_2), \quad a: c_1 \to c_2 \quad \text{and} \quad u: x_1 \to f(a)x_2.$$

Then $\Pi_f: f * C \to C$ is given by $\Pi_f((c, x)) = c$ on objects and $\Pi_f((a, u) = a$ on arrows.

Next, we will describe the natural transformations $\eta: \mathrm{Id} \to *D$ and $\xi: D* \to \mathrm{Id}$. Suppose $F: X \to C$ is an object in $\mathrm{CAT}\downarrow C$. Then $\eta_F: X \to DF * C$ is given by $\eta_F(x) = (F(x), (\mathrm{id}_{F(x)}, x))$ on objects and if $a: x_1 \to x_2$ is an arrow in X, then

$$\eta_F(a: x_1 \to x_2) = (F(a), a):(F(x_1), (\mathrm{id}_{F(x)}, x_1)) \to F(x_2), (\mathrm{id}_{F(x)}, x_2))$$

on arrows. It is trivial to check that $\Pi_F \eta_F = F$.

Now, suppose $f \in \mathrm{Obj}(\mathrm{CAT}^{C^{\mathrm{op}}})$. Then $\xi_f: D(f * C) \to f$ is given by $\xi_f(c)$: $D(f * C)(c) \to f(c)$ where $\xi_f(c)(a, (c', x)) = f(a)x$ on $\mathrm{Obj}(D(f * C)(c))$ and $\xi_f(c)(a, u):(a_1, (c'_1, x_1)) \to (a_2, (c'_2, x_2)) = f(a_1)u: f(a_1)x_1 \to f(a_2)x_2$ on arrows.

0.3. *The classifying space functor, homotopy and (co)homology*

There is a well-known functor $| |: \mathrm{SET}^{\mathrm{Ord}} \to \mathrm{TOP}$, called geometric realization, which takes simplicial sets (actually CW-complexes) to topological spaces. Suppose $S: \mathrm{Ord}^{\mathrm{op}} \to \mathrm{SET}$ is a given simplicial set. Then

$$|S| = U(\Delta^n \times S(n))/\sim,$$

where $\sim$ is given by $(\Delta(a)(x), u) \sim (x, S(a)(u))$ for $a: n \to m$ in Ord, $x \in \Delta^n$, and $u \in S(m)$.

Now, given a category C, we have a naturally associated simplicial set $\mathrm{NC}: \mathrm{Ord}^{\mathrm{op}} \to \mathrm{SET}$ given by $\mathrm{NC}(n) = \mathrm{Funct}(n, C)$, the set of all functors from n, where n is viewed as the totally ordered set of n elements, into C (see [8]). More geometrically, $\mathrm{NC}(n)$ can be viewed as the set of all commutative "n-simplices" of arrows in C.

Example 0.3.1. An element of NC(3) could be depicted diagramatically as follows:

$$c_0 \xrightarrow{a_1} c_1 \xrightarrow{a_2} c_2 \xrightarrow{a_3} c_3$$

Notation. An element,

$$c_0 \xrightarrow{a_1} c_1 \xrightarrow{a_2} c_2 \cdots c_{n-1} \xrightarrow{a_n} c_n,$$

of $\mathrm{NC}(n)$ will be denoted from here on by $c_0[a_1, a_2, \ldots, a_n]$.

Note. We will denote the space $|\mathrm{NC}|$ by $|C|$.

Now, since $|C|$ is a CW-complex, its singular homology is isomorphic to its cellular homology, which is given by the following chain complex:

$$\partial_n : Z(|C|_n) \to Z(|C|_{n-1}),$$

$$\partial_n(c_0[a_1, a_2, \dots, a_n]) = c_1[a_2, \dots, a_n] + \sum_{i=1}^{n-1} (-1)^i c_0[a_1, \dots, a_i a_{i+1}, \dots, a_n]$$
$$+ (-1)^n c_0[a_1, \dots, a_{n-1}],$$

where $Z(|C|_n)$ denotes the free abelian group on the n-cells of $|C|$.

Example 0.3.2. If $f : M \to N$ is a monoid homomorphism, then N is right M-set with the obvious action. We hence may form $N//M$. This category has an N^{op} action by left N multiplication on the objects. It is easy to see that this is Df where D is as in the GKNT adjunction. The homology of $|Df|$ is given by the homology of the chain complex $\{ZN(N//M)_*, \partial_*\}$ where $ZN(N//M)_p$ is the free abelian group generated by elements of the form $n[m_1, m_2, \dots, m_p]$. In particular, for $I : M \to M$, the identity, the cellular homology chain complex associated with $|DI|$, is just the standard bar resolution.

Example 0.3.3. First consider $U_1 = \{1, e\}$ where $e^2 = e$. Then $\pi_1(|U_1|)$ is determined by the 2-skeleton of $|U_1|$. In addition, the loop associated with e is the only possible generator for $\pi_1(|U_1|)$. But this loop is bound by the disk determined by the 2-simplex associated with $[e, e]$. Hence, $\pi_1(|U_1|) = 0$.

I claim that $|U_1|$ is contractible. By the Hurewicz Theorem, it is enough to show that $|U_1|$ is acyclic. Denote by $\{Z|U_1|_p, \partial_p\}$ the chain complex giving the cellular homology of $|U_1|$. Then it is easy to check that $D_{p+1} : Z|U_1| \to Z|U_1|_{p+1}$ given by

$$D_{p+1}([e_1, \dots, e_p]) = (-1)^{p+1}[e_1, \dots, e_p, e_{p+1}],$$

where $e_i = e$ for all i, is a chain homotopy from the identity to the zero map.

Example 0.3.4. *An aperiodic* monoid, M, is a monoid such that for ever $m \in M$, there exists $i \in Z^+$ such that $m^{i+1} = m^i$. Now, every 1-cell in M is indexed by some $m \in M$. But then this 1-cell will bound the 2-cell indexed by $1[m^i, m]$ where $m^{i+1} = m^i$. Thus we see that aperiodic monoids have simply connected classifying spaces.

Now, in the case where C is a group, we have that $|C|$ is a $K(C, 1)$ space. Thus, from the point of view of homotopy theory, $|C|$ is not very interesting to study. On the other hand, even when c is a monoid, $|C|$ is not in general $K(\pi, 1)$. In fact, in [4] it is shown that every Δ-set (a simplicial set without degeneracies). X, has the property that $|X|$ is homotopy equivalent to the classifying space of a discrete monoid!

We now proceed to describe some of the basic results in the homotopy theory of classifying spaces.

Lemma 0.3.5 (Segal [8]). *A natural transformation* $\eta : F_0 \to F_1$ *between functors* $F_i : X \to Y$, $i = 0, 1$, *induces a homotopy*

$$|\eta| : |X| \times I \to |Y|$$

such that $|\eta|(x, i) = |F_i|(x)$ *for* $i = 0, 1$.

Proof. We follow Quillen's proof. Consider the category 1 given by the following diagram:

$$0 \longrightarrow 1.$$

Next define a functor $G : X \times 1 \to Y$ given by $G(x, i) = F_i(x)$ on objects and

$$G(a : x_1 \to x_2, \mathrm{id} : i \to i) = F_i(a : x_1 \to x_2), \quad i = 0, 1,$$

$$G(a : x_1 \to x_2, 0 \to 1) = \eta(x_2)F_0(a) = F_1(a)\eta(x_1).$$

The last equation holds since η is a natural transformation. It also implies that G is a functor. Then it is known that $|1| \approx_{\mathrm{h}} I$ and that $|X \times Z| \approx_{\mathrm{h}} |X| \times |Z|$. On then easily checks that G is the desired homotopy. □

Remark. In the case of monoid homomorphisms $f_i : M \to N$, recall that a natural transformation $\eta : f_0 \to f_1$ amounts to an element $n \in N$ such that $f_1(m)n = nf_2(m)$ for all $m \in M$.

Corollary 0.3.6 (Segal [8]). *If* $F : X \to Y$ *has an adjoint, then* $|F|$ *is a homotopy equivalence.*

Corollary 0.3.7 (Segal [8]). *Categories with punctual objects have contractible classifying spaces.*

We now apply Corollary 0.3.6 and the above remark to obtain the following proposition.

Proposition 0.3.8. *Suppose* M *is a monoid with a zero. Then* $|M|$ *is contractible.*

Proof. Consider the obvious morphism $f : M \to U_1$. By the above example and corollary, it will be enough to show that f has an adjoint. Denote by z a zero in M. Then we have an injection $g : U_1 \to M$ given by $g(1) = 1$ and $g(e) = z$. It is clear that z provides the natural transformation $z : \mathrm{Id} \to gf$. Also $fg = \mathrm{Id}$ and thus f has an adjoint. □

Applying the above corollary to the GKNT adjunction, we also obtain Proposition 0.3.9.

Proposition 0.3.9. *Let* $F : X \to Y$ *be a functor. Then* $|\eta_F| : |X| \to |DF * Y|$ *is a homotopy equivalence.*

Proof. Recall that $\eta_F : X \to DF * Y$ is given by $\eta_F(x) = (F(x), (\mathrm{id}_x, x))$ on objects and by $\eta_F(a) = (F(a), a)$ on arrows.

Define $G : DF * Y \to X$ by $G(y, (a, x)) = x$ on objects and $G((b, a) : (y_1, (b_1, x_1)) \to (y_2, (b_2, x_2)) = a : x_1 \to x_2$. This is trivially a functor. Furthermore, $G\eta_F = \mathrm{ID} : X \to X$.

Now, define $T : \mathrm{ID} \to \eta_F G$ by

$$T(y, (b, x)) = (b, \mathrm{id}_x) : (y, (b, x)) \to (F(x), (\mathrm{id}, x)).$$

We must show that T is indeed a natural transformation. Suppose $(b, a) : (y_1, (b_1, x_1)) \to (y_2, (b_2, x_2))$ is an arrow in $DF * Y$. Then

$$T(y_2, (b_2, x_2))\,(b, a) = (b_2, \mathrm{id}_{x_2})\,(b, a) = (b_2 b, \mathrm{id}_{x_2} DF(b_2) a).$$

On the other hand,

$$\eta_F G(b, a)\, T(y_1, (b_1, x_1)) = (F(a), a)\,(b_1, \mathrm{id}_{x_1}) = F((a) b_1, a DF(F(a)) \mathrm{id}_{x_1}).$$

The following commutative diagram then shows that

$$T(y_2, (b_2, x_2))\,(b, a) = \eta_F G(b, a)\, T(y_1, (b_1, x_1)):$$

$$\begin{tikzcd}
y_1 \arrow[r, "b_1"] \arrow[d, "b"'] & F(x_1) \arrow[d, "F(a)"] \\
y_2 \arrow[r, "b_2"] & F(x_2)
\end{tikzcd}$$

We are then done by Corollary 0.3.6. □

There is also a dual result for $\mathrm{CAT}^{C^{\mathrm{op}}}$.

Proposition 0.3.10. *Let $f : C^{\mathrm{op}} \to \mathrm{CAT}$ be an object in $\mathrm{CAT}^{C^{\mathrm{op}}}$. Then $|\xi_f(c)| : |D(f * C)\,(c)| \to |f(c)|$ is a homotopy equivalence for all $c \in \mathrm{Obj}(C)$.*

Proof. Recall that $\xi_f(c) : D(f * C)\,(c) \to f(c)$ is given by $\xi_f(c)(a, (c', u)) = f(a) u$ on objects and $\xi_f(c)\,((a, b) : (a_1, c' u_1)) \to (a_2, (c'_2, u_2))) = f(a_1) b : f(a_1) u_1 \to f(a_2) u_2$ on arrows. Define $G_c : f(c) \to D(f * C)\,(c)$ by $G_c(u) = (\mathrm{id}_c, (c, u))$ on objects and $G_c(b : u_1 \to u_2) = (\mathrm{id}_c, b); (\mathrm{id}_c, (c, u_1)) \to (\mathrm{id}_c, (c, u_2))$ on arrows. It is clear that this is a functor. Also, $\xi_f(c) G_c = \mathrm{Id} : f(c) \to f(c)$. Now, $G_c \xi_f(c) : D(f * C)\,(c) \to D(f * C)\,(c)$ is given by

$$G_c \xi_f(c)\,((a, (c', u))) = (\mathrm{id}_c, (c, f(a) u)) \quad \text{on objects},$$

$$G_c \xi_f(c)\,((a, b)) = (\mathrm{id}_c, f(a)\, b) \qquad \text{on arrows}.$$

Define $S_c : G_c \xi_f(c) \to \mathrm{Id}$ by

$$S_c((a, (c', u))) = (a, \mathrm{Id}_{f(a) u}) : (\mathrm{id}_c, (c, f(a) u)) \to (a, (c', u)).$$

We show that S_c is a natural transformation. Suppose that

$$(a, b):(a_1, (c'_1, u_1)) \to (a_2, (c'_2, u_2))$$

is an arrow in $D(f * C)(c)$.

Then we must show that the following diagram is commutative:

$$\begin{tikzcd}
(\mathrm{Id}_c, (c, f(a_1)u_1)) \arrow[r, "(a_1, \mathrm{Id}_{f(a_1)u_1})"] \arrow[d, "(\mathrm{Id}_c, f(a_2)b)"'] & (a_1, (c'_1, u_1)) \arrow[d, "(a, b)"] \\
(\mathrm{Id}_c, (c, f(a_2)u_2)) \arrow[r, "(a_2, \mathrm{Id}_{f(a_2)u_2})"] & (a_2, (c'_2, u_2))
\end{tikzcd}$$

But since we have $a_1 a = a_2$, we see that

$$(a_1, \mathrm{Id}_{f(a_1)u_1})(a, b) = (a_1 a, \mathrm{Id}_{f(a_1)u_1} f(a_1)b) = (a_2, f(a_1)b).$$

On the other hand,

$$(\mathrm{Id}_c, f(a_1)b)(a_2, \mathrm{Id}_{f(a_2)u_2}) = (a_2, f(a_1)b)$$

and the diagram commutes. Thus S_c is a natural transformation and we are done by Corollary 0.3.6. □

Let $F: Y \to X$ be a functor. By the GKNT adjunction, $DF: X^{\mathrm{op}} \to \mathrm{CAT}$ is given by $DF(y) = y{\downarrow}F$ on objects. We then have the following due to Quillen.

Theorem 0.3.11 (Quillen [6]). *Suppose $F: X \to Y$ is such that for every arrow $a: y_1 \to y_2$ in Y, $|a|: |y_2{\downarrow}F| \to |y_1{\downarrow}F| \to |y_1{\downarrow}F|$ is a homotopy equivalence. Then there is along exact sequence of homotopy groups:*

$$\cdots \to \pi_{i+1}(|y{\downarrow}F|) \to \pi_{i+1}(|X|) \to \pi_{i+1}(|Y|) \to \pi_i(|y{\downarrow}F|) \to \cdots$$

Corollary 0.3.12. (Quillen [6]). *If $|y{\downarrow}F|$ is contractible for all $y \in \mathrm{Obj}(Y)$, then $|F|$ is a homotopy equivalence.*

Remark. Suppose $f: M \to N$ is a monoid homomorphism. Then note that $Df = N//M$, equipped with the obvious contravariant action by N. The hypotheses of the above theorem do not always hold in this case. Consider $f: M \to N$ where $M = \{1, a_1, a_2\}$ with $a_i a_j = a_i$ and $N = \{1, e\}$ with $e^2 = e$ and $f(a_i) = e$. Then, since M and N consist of idempotents, it is easy to see that $\pi_1(|M|) = \pi_1(|N|) = 0$. On the other hand, by a simple long exact sequence argument applied to $|M| \subseteq |Df|$, it is easy to see that $\pi_1(|Df|) = Z$. Since $|N|$ is contractible by the above example, we see that the conclusion of the above theorem does not hold.

However, we do have the following proposition.

Proposition 0.3.13. *Suppose $F:X \to G$ is a functor where G is a group. Then there is a long exact sequence;*

$$\cdots \to \pi_{i+1}(|DF|) \to \pi_{i+1}(|X|) \to \pi_{i+1}(|G|) \to \pi_i(|DF|) \to \cdots.$$

Proof. First note that in this case DF is the category given by $\mathrm{Obj}(DF) = Gx\,\mathrm{Obj}(X)$ and $a:(g_1, x_1) \to (g_2, x_2)$ iff $a:x_1 \to x_2$ and $g_2 = g_1F(a)$.

Now, suppose $DF(g):DF \to DF$ is given. Then consider the comma category $(g', x)\downarrow DF(g)$. Since G is a group, we get that there exists a unique g'' such that $g' = gg''$. Now we claim that $(\mathrm{id}_x, (g'', x))$ is initial in $(g', x)\downarrow DF(g)$. Suppose $(a, (h, y))$ is an object in $(g', x)\downarrow DF(g)$. Then $a:x \to y$ and $g'F(a) = gg''F(a) = gh$. But this implies that $g''F(a) = h$ and we get an arrow $a:(\mathrm{id}_x, (g'', x)) \to (a, (h, x))$. It is clear that it is also the *only arrow* from $(\mathrm{id}_x, (g'', x))$ to $(a, (h, x))$. We are then done by Corollaries 0.3.12 and 0.3.7. □

There is a dual theorem to Theorem 0.3.11.

Proposition 0.3.14. *Suppose $F: Y^{\mathrm{op}} \to \mathrm{CAT}$ is such that for every $a:y_1 \to y_2$ in Y, $|F(a)|:|F(y_2)| \to |F(y_1)|$ is a homotopy equivalence then we have a long exact sequence:*

$$\cdots \to \pi_i(|F(y)|) \to \pi_1(|F*Y|) \to \pi_i(|Y|) \to \pi_{i-1}(|F(y)|) \to \cdots.$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
D(F*Y)(y_2) \arrow[r, "F(y_2)"] \arrow[d, "D(F*Y)(a)"'] & F(y_2) \arrow[d, "F(a)"] \\
D(F*Y)(y_1) \arrow[r, "F(y_1)"'] & F(y_1)
\end{tikzcd}$$

By proposition 0.3.10, $|D(F*Y)(a)|$ is a homotopy equivalence iff $|F(a)|$ is. The proposition then follows from Theorem 0.3.11 and Proposition 0.3.10. □

Corollary 0.3.15. *Denote by $M \circ N$ the* wreath product of M with N. *Then if $F:N \to \mathrm{End}(\mathrm{Map}(N, M))$ is such that*

$$|F(a)|:|\mathrm{Map}(N, M)| \to |\mathrm{Map}(N, M)|$$

is a homotopy equivalence for every $a \in N$, then we have a long exact sequence:

$$\cdots \to \bigoplus \pi_{i+1}(|M|) \to \pi_{i+1}(|M \circ N|) \to \pi_{i+1}(|N|) \to \bigoplus \pi_i(|M|) \to \cdots,$$

where $\bigoplus \pi_i(M|)$ denotes $o(N)$ copies of $\pi_i(|M|)$ ($o(N)$ is the order of N).

Proof. The wreath product $M \circ N$ is easily seen to be $F*N$. If N acts through homotopy equivalences on $\mathrm{Map}(N, M)$, then by Proposition 0.3.14, we obtain a long exact sequence:

$$\cdots \to \pi_i(|\mathrm{Map}(N, M)|) \to \pi_i(|M \circ N|) \to \pi_i(|N|) \to \pi_{i-1}(|\mathrm{Map}(N, M)|) \to \cdots .$$

But then $\mathrm{Map}(N, M) = \bigoplus M$. □

Example 0.3.16. Suppose $|M|$ is contractible. Then, since classifying spaces of discrete monoids are CW-complexes, we see by the above corollary that $|M \circ N|$ is homotopy equivalent to $|N|$. For example, we have that $|U_1 \circ M| \simeq |M|$.

Example 0.3.17. Given a group G and a monoid M, there is *always* a long exact sequence:

$$\cdots \to \bigoplus \pi_{i+1}(|M|) \to \pi_{i+1}(|M \circ G|) \to \pi_{i+1}(|G|) \to \bigoplus \pi_i(|M|) \to \cdots,$$

where $\bigoplus \pi_i(|M|)$ denotes the K-fold direct sum of $\pi_i(|M|)$ and K is the order of G. This is clear from Propositions 0.3.13 and 0.3.10. But $|G|$ is a $K(G, 1)$ space and we obtain a short exact sequence:

$$0 \to \pi_i(|M|) \to \pi_i(|M \circ G|) \to \pi(|G|) \to 0$$

and isomorphisms $\bigoplus \pi_i(M) \simeq \pi_i(|M \circ G|)$ for $i > 1$. In particular, note that $|M \circ G|$ is a $K(G, 1)$ space if $|M|$ is contractible.

It should be clear by now that at least the (co)homology theory of $|C|$ really had nothing to do with the actual topology of $|C|$. Indeed, the homology of C is given by the chain complex $\{\mathrm{ZNC}_i, \partial_i\}$ associated with the simplicial set NC. This suggests that we define the homology of a given category C without reference to $|C|$. This can be done as follows: Suppose $F: C \to AB$ is given. Then the homology of C with coefficients in F is the homology of the following chain complex:

$$\left\{ A(C, F)_p = \bigoplus_{c_0[a_1, a_2, \ldots, a_p]} F(c_0), \partial_p \right\},$$

with $\partial_p: A(C, F)_p \to A(C, F)_p$ given by

$$\begin{aligned} \partial_p(x, c_0[a_1, a_2, \ldots, a_p]) = {} & (F(a_1)x, c_1[a_2, a_3, \ldots, a_p]) \\ & + \sum_{i=1}^{p-1} (-1)^i (x, c_0[a_1, \ldots, a_i a_{i+1}, \ldots, a_p]) \\ & + (-1)^p (x, c_0[a_1, \ldots, a_{p-1}]). \end{aligned}$$

We will denote this homology theory by $H_*(C, F)$. In particular, one easily sees that in the case where C is a monoid, then $F: M \to AB$ is just a right M-module and we have the standard homology theory of monoids.

0.4. The (co)homology of categories as a triple (co)homology

Recall that adjunctions (or comonads associated with adjunctions) give rise to simplicial objects in functor categories (see [5]). In the case of the GKNT adjunction, this amounts to a simplicial object in A^A, where $A = \mathrm{CAT}^{C^{\mathrm{op}}}$. The sequence of endofunctors in the simplicial object is $\{L^n\}$ where $L^n = (D*)^n$ and, by evaluation on an object in $\mathrm{CAT}^{C^{\mathrm{op}}}$, we obtain a simplicial object in $\mathrm{CAT}^{C^{\mathrm{op}}}$.

Let us consider the case where we evaluate L^n at $D(\mathrm{Id}_C)$. Then it is not difficult to see that $L^n(D(\mathrm{Id}_C)): C^{\mathrm{op}} \to \mathrm{CAT}$ is given by the following:

$$\begin{aligned}\mathrm{Obj}(L^{n-1}(D(\mathrm{Id}_C))(c)) &= c \xrightarrow{a_1} c \xrightarrow{a_2} c \xrightarrow{a_3} \cdots \xrightarrow{a_n} c \\ &= c[a_1, a_2, \ldots, a_n].\end{aligned}$$

An arrow between

$$c \xrightarrow{a_1} c_1 \xrightarrow{a_2} c_2 \cdots \xrightarrow{a_n} c_n$$

and

$$c \xrightarrow{b_1} d_1 \xrightarrow{b_2} d_2 \cdots \xrightarrow{b_n} d_n$$

is a diagram of the following form:

$$\begin{tikzcd} & c_1 \arrow[r, "a_2"] \arrow[dd, "e_1"] & c_2 \arrow[r, "a_3"] \arrow[dd, "e_2"] & c_3 \cdots \arrow[r, "a_n"] \arrow[dd, "e_3"] & c_n \arrow[dd, "e_n"] \\ c \arrow[ur, "a_1"] \arrow[dr, "b_1"'] & & & & \\ & d_1 \arrow[r, "b_2"'] & d_2 \arrow[r, "b_3"'] & d_3 \cdots \arrow[r, "b_n"'] & d_n \end{tikzcd}$$

Then if $a: c' \to c$ is an arrow in C,

$$L^{n-1}(D(\mathrm{Id}_C))(a): L^{n-1}(D(\mathrm{Id}_C))(c) \to L^{n-1}(D(\mathrm{Id}_C))(c')$$

is given by $c[a_1, \ldots, a_n] \to c'[aa_1, \ldots, a_n]$ on objects and the labels on the arrows do not change.

Next, following the notation on p. 177 of [5], we see that

$$d_i(D(\mathrm{Id}_C)): L^n(D(\mathrm{Id}_C)) \to L^{n-1}(D(\mathrm{Id}_C))$$

is given by

$$d_i(D(\mathrm{Id}_C))(c[a_1, \ldots, a_{n+1}]) = c[a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}]$$

on objects.

Now let $F: \mathrm{CAT} \to \mathrm{SET}$ and $Z: \mathrm{SET} \to AB$ denote the forgetful functor on objects and the free abelian group functor respectively. Then we have $(ZF)^{C^{\mathrm{op}}}$:

$\mathrm{CAT}^{C^{\mathrm{op}}} \to AB^{C^{\mathrm{op}}}$. Since $AB^{C^{\mathrm{op}}}$ is an AB-category, we can set

$$\partial A = \sum_{i=1}^{n} (-1)^i d_i(D(\mathrm{Id}_C)) : (ZF)^{C^{\mathrm{op}}}(L^n(D(\mathrm{Id}_C)) \longrightarrow (ZF)^{C^{\mathrm{op}}}(L^{n-1}(D(\mathrm{Id}_C))$$

and get a resolution in the sense of homological algebra.

Define the *tensor product functor over C* as follows:

$$\otimes_C : AB^C \times AB^{C^{\mathrm{op}}} \longrightarrow AB$$

to be $F \otimes_C G = Z(F(c) \times G(c))/\sim$ where $(xa, y) \sim (x, ay)$ for $x \in F(c_1)$, $y \in G(c_2)$, and $a : c_1 \to c_2$. It is easy to check that $\otimes_C$ is a bifunctor. In particular, note that if $F \in \mathrm{Obj}(AB^C)$ is given, then $F \otimes_C (\ ) : AB^{C^{\mathrm{op}}} \to AB$ is a functor.

Given such an F, we define *the GKNT homology of C with coefficients in F*, $K_*(C, F)$, to be the homology of the chain complex $\{F \otimes_C (ZF)^{C^{\mathrm{op}}}(L^n(D(\mathrm{Id})), F \otimes_C \partial^A\}$. We then have the following theorem.

Theorem 0.4.1. $H_*(C, F) = K_*(C, F)$.

Proof. We will show that the chain complexes are isomorphic.

First, define $g_n : F \otimes_C (ZF)^{C^{\mathrm{op}}}(L^n(D(\mathrm{Id}_C)) \to A_n(C, F)$ by $g_n(u \otimes_C c[a_1, \dots, a_{n+1}]) = (ua_1, c_1[a_2, \dots, a_{n+1}])$. It is trivial to check that this is well defined. I claim that this is a chain map of degree 0.

$$\partial_n g_n(u \otimes_C c[a_1, \dots, a_{n+1}]) = (ua_1a_2, c_2[a_3, \dots, a_{n+1}])$$
$$+ \sum_{i=2}^{n} (-1)^i (ua_1, c_1[a_2, \dots, a_ia_{i+1}, \dots, a_{n+1}]) + (-1)^{n+1}(ua_1, c_1[a_2, \dots, a_n].$$

On the other hand,

$$\begin{aligned}
&g_{n-1}F \otimes_C \partial_n^A(u \otimes_C c[a_1, \dots, a_{n+1}]) \\
&\quad = g_{n-1}(u \otimes_C c[a_1a_2, \dots, a_{n+1}]) \\
&\qquad + \sum_{i=2}^{n} (-1)^i g_{n-1}(u \otimes_C c[a_1, \dots, a_ia_{i+1}, \dots, a_{n+1}]) \\
&\qquad + (-1)^{n+1} g_{n-1}(u \otimes_C c[a_1, \dots, a_n]) \\
&\quad = g_{n-1}(ua_1 \otimes_C c_1[a_2, \dots, a_{n+1}]) \\
&\qquad + \sum_{i=2}^{n} (-1)^i g_{n-1}(u \otimes_C c[a_1, \dots, a_ia_{i+1}, \dots, a_{n+1}]) \\
&\qquad + (-1)^{n+1} g_{n-1}(u \otimes_C c[a_1, \dots, a_n]) \\
&\quad = \partial_n g_n(u \otimes_C c[a_1, \dots, a_{n+1}]).
\end{aligned}$$

Thus g_* is a chain map of degree 0.

Then, define $h_n : A_n(C, F) \to F \otimes_C (ZF)^{C^{op}} (L^n(D(\mathrm{Id}_C))$ by

$$h_n(u, c[a_1, \dots, a_n]) = u \otimes_C c[1, a_1, \dots, a_n]).$$

We then have

$$\begin{aligned} F \otimes_C \partial^A h_n(u, c[a_1, \dots, a_n]) = {} & u \otimes_C c[a_1, \dots, a_n]) \\ & + \sum_{i=1}^{n-1} (-1)^i (u \otimes_C c[1, a_1, \dots, a_i a_{i+1}, \dots, a_n]) \\ & + (-1)^n (u \otimes_C c[1, a_1, \dots, a_{n-1}]). \end{aligned}$$

On the other hand,

$$\begin{aligned} h_{n-1} \partial_n (u, c[a_1, \dots, a_n]) = {} & h_{n-1}(ua_1 \otimes_C c_1[a_2, \dots, a_n]) \\ & + \sum_{i=1}^{n-1} (-1)^i h_{n-1}(u \otimes_C c[a_1, \dots, a_i a_{i+1}, \dots, a_n]) \\ & + (-1)^n h_{n-1}(u \otimes_C c[a_1, \dots, a_{n-1}]). \end{aligned}$$

Then, $h_n g_n : F \otimes_C (ZF)^{C^{op}} (L^n(D(\mathrm{Id}_C)) \to F \otimes_C (ZF)^{C^{op}} (L^n(D(\mathrm{Id}_C))$ is given by

$$\begin{aligned} g_n(u \otimes_C c[a_1, \dots, a_{n+1}]) &= ua_1 \otimes_C c[1, a_1, \dots, a_{n+1}] \\ &= u \otimes_C c[a_1, a_2, \dots, a_{n+1}] \end{aligned}$$

by the definition of the tensor product.

Also, $g_n h_n : A_n(C, F) \to A_n(C, F)$ is given by

$$g_n h_n(u, c[a_1, \dots, a_n]) = g_n(u \otimes_C c[1, a_1, \dots, a_n]) = u \otimes_C c[a_1, \dots, a_n].$$

This concludes the proof. $\square$

Now consider the following coefficients for C: $F_C : C \to AB$ given by $F_C(c) = Z$ for every $c \in \mathrm{Obj}(C)$ and if $a : c_1 \to c_2$, then $F_C(a) = \mathrm{Id} : Z \to Z$ maps generator to generator. We then have the following proposition.

Proposition 0.4.2. $H_*(|C|, Z) = K_*(C, F_C)$.

Remark. We also for every $F \in \mathrm{Obj}(AB^{C^{op}})$ an associated functor $\mathrm{Hom}(\ , F) : AB^{C^{op}} \to AB$. Then *the GNKT cohomology of C with coefficients in F, $K*(C, F)$*, is the homology of the chain complex $\{\mathrm{Hom}((ZF)^{C^{op}}(L^n(D(\mathrm{Id}_C)), F)),$ $\mathrm{Hom}(\partial_n^A, F)\}$.

We would like to make a connection between cohomology with local coefficients in the sense of classical algebraic topology and GKNT co-homology.

Let X be a topological space and $\Pi_1(X)$ denote the fundamental groupoid of X. Then a *system of local coefficients for X* is by definition a functor $F : \Pi_1(X) \to AB$. Denote by $C^n(X, F)$ the set of all functions c which assign, to each singular simplex

$u: \Delta^n \to X$, an element $c(u) \in F(e_0)$). Then $C^n(X, F)$ becomes a group under addition of functional values. We define the coboundary operator $\delta^{n+1}: C^n(X, F) \to C^{n+1}(X, F)$ by

$$(-1)^n \delta^n(c)(u) = F(u_1)^{-1} c(\partial_0 u) + \sum_{i=1}^{n} (-1)^i c(\partial_i u).$$

(Here u_1 denotes the equivalence class of $u([e_0, e_1])$ in $\Pi_1(X)$ where $[e_0, e_1]$ is the "first" edge in δ^{n+1}.) It is easy to check that $\delta^{n+2}\delta^{n+1} = 0$, Then the *cohomology of X with local coefficients in F*, $H^*(X, F)$, is by definition the homology given by the above chain complex.

It is known (see [14] for instance) that in the case where X is CW-complex, then the cohomology $H^*(X, F)$ can be computed from the CW-structure equipped with the pull-back local coefficient structure on each n-cell. We will describe this in the case where $X = |C|$. We will also denote by $(c_0[a_1, \ldots, a_n])$ the n-cell in $|C|$ indexed by $c_0[a_1, \ldots, a_n]$.

Now, for a given category C, note that we have a functor $p_C: C \to \Pi_1(C)$ given by $p_C(c) = c \in |C|_0 \subseteq |C|$ on objects and $p_C(a: c_1 \to c_2)$ is the equivalence class of the path determined by a in $|C|$. We also have a corresponding opposite functor $C^{\mathrm{op}}: C^{\mathrm{op}} \to \Pi_1(C)^{\mathrm{op}} \simeq \Pi_1(|C|)$. These functors induce:

$$AB^{p_C}: AB^{\Pi_1(|C|)} \to AB^C,$$

$$AB^{p_{C^{\mathrm{op}}}}: AB^{\Pi_1(|C|)} \to AB^{C^{\mathrm{op}}}.$$

By a slight abuse of notation, if $F \in \mathrm{Obj}(AB^{\Pi_1(|C|)})$, we will denote by F and F^{-1} its image under the above two functors respectively.

Now, denote by $G^n(C, F)$ the set of all functions u that assign to each $(c_0[a_1, \ldots, a_n])$ an element of $F(c_0)$. Then we have a coboundary operator $\delta^{n+1}: G^n(C, F) \to G^{n+1}(C, F)$ given by

$$\begin{aligned}\delta^{n+1}(u)(c_0[a_1, \ldots, a_{n+1}]) = {} & F(a_1)^{-1}(u(c_1([a_2, \ldots, a_{n+1}])) \\ & + \sum_{i=1}^{n} (-1)^i u(c_0[a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}]) \\ & + (-1)^{n+1} u(c_0[a_1, \ldots, a_n]).\end{aligned}$$

Once again, it is easy to check that $\delta^{n+2}\delta^{n+1} = 0$. The homology groups of the above complex are isomorphic to $H^*(|C|, F)$.

We then obtain the dual theorem to Theorem 0.4.1.

Theorem 0.4.3. $H^*(|C|, F) \simeq K^*(C, F^{-1})$ *where the right-hand side is the GKNT cohomology of C with coefficients in* F^{-1}.

Proof. For simplicity of notation, we will denote by $L^n(C)$, $(ZF)^{C^{\mathrm{op}}}(L^n(D(\mathrm{Id}_C)) \in AB^{C^{\mathrm{op}}}$. Now to prove the theorem, we will show that $\{\mathrm{Hom}(L^*(C), F^{-1}), \delta^*\}$ is isomorphic to $\{G^*(C, F), \delta^*]$ as chain complexes.

First, define $g^n\colon G^n(C,F) \to \mathrm{Hom}(L^n(C), F^{-1})$ by the following formula:

$$g^n(u)(c_0[a_1, \dots, a_{n+1}]) = F(a_1)^{-1}u(c_1[a_2, \dots, a_{n+1}]).$$

First we must show that $g^n(u) \in \mathrm{Hom}(L^n(C), F^{-1})$. Suppose that $a\colon c_0' \to c_0$. Then

$$\begin{aligned} g^n(u)(L^n(C)(a)(c_0[a_1, \dots, a_n]) &= g^n(u)(c_0'[aa_1, a_2, \dots, a_{n+1}]) \\ &= F(aa_1)^{-1}u(c_1[a_2, \dots, a_{n+1}]) \\ &= F(a)^{-1}F(a_1)^{-1}u(c_1[a_2, \dots, a_{n+2}]) \\ &= F(a)^{-1}g^n(u)(c_0[a_1, \dots, a_{n+1}]). \end{aligned}$$

This shows that $g^n(u)$ is indeed a natural transformation from $L^n(C)$ to F^{-1}, i.e., an element of $\mathrm{Hom}(L^n(C), F^{-1})$. Next we claim that g^* is a chain map. First, we compute $\delta^{n+1}g^n(u)$ as follows:

$$\begin{aligned} \delta^{n+1}g^n(u)(c_0[a_1, \dots, a_{n+2}]) &= g^n(u)(\partial_{n+1}(c_0[a_1, \dots, a_{n+2}])) \\ &= \sum_{i=1}^{n+1}(-1)^i g^n(u)(c_0[a_1, \dots, a_ia_{i+1}, \dots, a_{n+2}]) \\ &\quad + (-1)^{n+2}g^n(u)(c_0[a_1, \dots, a_{n+1}]) \\ &= F^{-1}(a_1a_2)u(c_2[a_3, \dots, a_{n+2}]) \\ &\quad \times \sum_{i=1}^{n+1}(-1)^i F^{-1}(a_1)u(c_1[a_2, \dots, a_ia_{i+1}, \dots, a_{n+2}]) \\ &\quad \times (-1)^{n+2}F^{-1}(a_1)u(c_0[a_2, \dots, a_{n+1}]). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\delta^{n+1}\delta^{n+1}(u)(c_0[a_1, \dots, a_{n+2}]) \\ &\quad = F^{-1}(a_1)\delta^{n+1}(u)c_1[a_2, \dots, a_{n+2}]) \\ &\quad = F^{-1}(a_1)F^{-1}(a_2)u(c_2[a_2, \dots, a_{n+2}]) \\ &\quad\quad \times \sum_{i=2}^{n+1}(-1)^i F^{-1}(a_1)u(c_1[a_2, \dots, a_ia_{i+1}, \dots, a_{n+2}]) \\ &\quad\quad \times (-1)^{n+2}F^{-1}(a_1)u(c_0[a_2, \dots, a_{n+1}]). \end{aligned}$$

Therefore g^* is a chain map.

Next define $h^n\colon \mathrm{Hom}(L^n(C), F^{-1}) \to G^n(C,F)$ by the following formula:

$$h^n(v)(c_0[a_1, \dots, a_{n+1}]) = v(c_0[1, a_1, \dots, a_{n+1}]),$$

$$\begin{aligned} \delta^{n+1}h^n(v)(c_0[a_1, \dots, a_{n+2}]) &= F^{-1}(a_1)h^n(v)(c_1[a_2, \dots, a_{n+2}]) \\ &\quad + \sum_{i=1}^{n+1}(-1)^i h^n(v)(c_0[a_1, \dots, a_ia_{i+1}, \dots, a_{n+2}]) \end{aligned}$$

$$
\begin{aligned}
&+(-1)^{n+2}h^n(v)(c_0[a_1,\ldots,a_{n+1}])\\
&=F^{-1}(a_1)v(c_1[1,a_2,\ldots,a_{n+2}])\\
&\quad+\sum_{i=1}^{n+1}(-1)^i v(c_0[1,a_1,\ldots,a_ia_{i+1},\ldots,a_{n+2}])\\
&\quad+(-1)^{n+2}v(c_0[1,a_1,\ldots,a_{n+1}])\\
&\quad+(-1)^{n+2}v(c_0[1,a_1,\ldots,a_{n+1}])\\
&=v(c_0[a_1,a_2,\ldots,a_{n+2}])\\
&\quad+\sum_{i=1}^{n+1}(-1)^i v(c_0[1,a_1,\ldots,a_ia_{i+1},\ldots,a_{n+2}])\\
&\quad+(-1)^{n+2}v(c_0[1,a_1,\ldots,a_{n+1}]).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
h^{n+1}\delta^{n+1}(v)(c_0[a_1,\ldots,a_{n+2}]) &= \delta^{n+1}(v)(c_0[1,a_1,\ldots,a_{n+2}])\\
&= v(c_0[a_1,a_2,\ldots,a_{n+2}])\\
&\quad+\sum_{i=1}^{n+1}(-1)^i v(c_0[1,a_1,\ldots,a_ia_{i+1},\ldots,a_{n+2}])\\
&\quad+(-1)^{n+2}v(c_0[1,a_1,\ldots,a_{n+1}]).
\end{aligned}
$$

Thus h^* is a chain map. Finally,

$$
\begin{aligned}
h^ng^n(u)(c_0[a_1,\ldots,a_{n+1}]) &= g^n(u)(c_0[1,a_1,\ldots,a_{n+1}])\\
&= F^{-1}(1)u(c_0[a_1,\ldots,a_{n+1}])\\
&= u(c_0[a_1,\ldots,a_{n+1}]). \qquad \square
\end{aligned}
$$

Now let $F:\Pi_1(M)\to AB$ be a local coefficient system for M. Let $*$ denote the object of M. Then $F(*)$ becomes a right and left $\Pi_1(M)$ module. Thus, it pulls back to become an M-module.

The following is a generalization of [14, Theorem 3.5*].

Corollary 0.4.4. $H^q(|M|,F(*))\simeq \mathrm{Ext}^q_{ZM}(Z,F(*))$.

Proof. Note that $L^n(M)$ is the standard bar resolution of Z by ZM-modules. Then apply Theorem 0.4.3. $\square$

Denote by $Z:\mathrm{SET}\to AB$ the free abelian group functor. For a monoid M we then have $Z^M:\mathrm{SET}^M\to AB^M$. We interpret this as M-sets get mapped to M-modules. On the other hand, we have $//M:\mathrm{SET}^M\to \mathrm{CAT}$ given by $X\to X//M$. The following lemma, although easy, is central to what follows.

Key Lemma. $H_i(M, ZX) = H_i(|X//M|, Z)$ *for* $i > 0$.

Proof. The left-hand side is the homology of the chain complex $\{ZX \otimes_{ZM} X_i, \partial_i\}$ where X_i denotes the standard bar resolution of M. On the other hand, the right-hand side is the homology of the chain complex $\{Z(|X//M|_i), \partial_i\}$ where $Z(|X//M|_i)$ is the free abelian group on $x[m_1, \dots, m_i]$.

Now define the following morphism,

$$f_i : ZX \otimes_{ZM} X_i \longrightarrow Z(|X//M|_i),$$

by $f_i(x \otimes_{ZM} m[m_1, \dots, m_i]) = f(xm \otimes_{ZM} [m_1, \dots, m_i]) = xm[m_1, \dots, m_i]$. It is trivial to check that this is well defined and gives rise to a 0-degree chain map $f_* : ZX \otimes_{ZM} X_* \to Z(|X//M|_*)$.

The inverse of f_* is given by $g_i : Z(|X//M|_i) \to ZX \otimes_{ZM} X_i$ where $g_i(x[m_1, \dots, m_i]) = x \otimes_{ZM} 1[m_1, \dots, m_i]$. □

0.5. *Finiteness conditions*

We will be interested in conditions under which a finite monoid M has the property that $|M|$ is homotopy equivalent to a finite CW-complex.

Definition 0.5.1. We say that a space X satisfies D_n iff the following two conditions hold:

(a) Let $\tilde{X}$ denote the universal cover of X. Then

$$H_i(\tilde{X}, Z) = 0 \quad \text{if } i > n.$$

(b) For all local coefficients F, $H^{n+1}(X, F) = 0$.

We then have the following theorem stated without proof.

Theorem 0.5.2. *Let $n > 3$. A space X is homotopy equivalent to a CW-complex of dimension $< n$ iff X satisfies D_n.*

Proof. See [13, p. 152]. □

Now, in the case of a finite monoid, M, we have the following lemma.

Lemma 0.5.3. *Let $\pi_1 : M \to \pi_1(|M|)$ be the natural morphism. Then $P : |\pi_1(|M|)//M| \to |M|$ is the universal cover of M.*

Proof. All that is needed to prove is that $\pi_1(|M|)$ acts on $|\pi_1(|M|)//M|$ by covering transformations and that $|\pi_1(|M|)//M|$ is simply connected.

But the action of $\pi_1(|M|)$ on $\pi_1(|M|)//M$ induces an action of $\pi_1(|M|)$ on $|\pi_1(|M|)//M|$ which covers $|M|$. It is also trivial to see that P restricted to the k-skeleton evenly covers the k-skeleton, so $P: |\pi_1(|M|)//M \to |M|$ is indeed a covering space.

To show that $|\pi_i(|M|)//M|$ is simply connected, note that by Proposition 0.3.13, we obtain a long exact sequence of homotopy groups:

$$\to \pi_i(|\pi_1(|M|)//M|) \to \pi_i(|M|) \to \pi_i(\pi_i(|M|)) \to \pi_{i-1}(|\pi_1(|M|)//M|) \to \cdots .$$

Now, note that $\pi_1(|\pi_1(|M|)|) = \pi_1(|M|)$ and that

$$\pi_1(|M|) \to \pi_1(|\pi_1(|M|)|)$$

is an isomorphism. Furthermore, $\pi_1(|M|)$ is a $K(\pi_1(|M|), 1)$ space. It follows that $|\pi_1(|M|)//M|$ is simply connected. □

Thus, using Lemma 0.5.3 and Theorem 0.4.3, we obtain the following proposition.

Proposition 0.5.4. *Let $n > 3$. Then $|M|$ is homotopy equivalent to a CW-complex of dimension $< n$ iff M satisfies the following two conditions*:

(1) $H_i(M, Z\pi_1(|M|)) = 0$ for i > n.

(2) $H^{n+1}(M, F) = 0$ *for all left* $\pi_1(|M|)$ *modules.*

Proof. (1) follows from Lemma 0.5.3 and the Key Lemma. (2) follows from Theorem 0.52 and Theorem 0.4.3. □

Notation. The above two conditions will be collectively called D'_n.

Corollary 0.5.5. *Suppose M has a simply connected classifying space. Then for $n > 3$, M is homotopy equivalent to a CW-complex with dimension $< n$ iff*

(1) $H_i(M, Z) = 0$ *for all* $i > n$,

(2) $H^{n+1}(M, Z) = 0$.

Proof. In this case $|M|$ is the universal cover of itself. Hence condition $D_n(a)$ becomes condition (1) above. Now, since $\pi_1(|M|) = 0$, we see that a left $\pi_1(|M|)$ module is simply an abelian group F. We are then done by the Universal Coefficient Theorem. □

Remark. Note that by Example 0.3.4, the above corollary applies to aperiodic monoids.

1. Spectral sequences in category and monoid theory

1.1. A spectral sequence associated with functors

Let $F: X \to Y$ be a functor between small categories. Then since $DF: Y^{\text{op}} \to \text{CAT}$, we obtain $|DF|: Y^{\text{op}} \to \text{TOP}$ given by $|DF|(y) = |DF(y)|$ on objects and if

$a: y_1 \to y_2$, then $|DF|(a) = |DF(a)|: |DF(y_2)| \to |DF(y_1)|$. We thus obtain a sequence of abelian functors $H_pDF: Y^{\text{op}} \to AB$ given by $H_pDF(y) = H_p(|DF(y)|, Z)$ on objects and if $a: y_1 \to y_2$, then $H_pDF(a) = H_p(|DF(a)|): H_p(|DF(y_2)|, Z) \to H_p(|DF(y_1)|, Z)$.

We then have the following theorem.

Theorem 1.1.1. *Given a functor $F: X \to Y$ between small categories, there is a spectral sequence whose $E^2_{p,q}$ term is $H_p(Y^{\text{op}}, H_qDF)$ and whose termination is $H_{p+q}(|X|, Z)$.*

The idea of the proof is derived from some constructions in [6].

Definition 1.1.2. A *bisimplicial set* is a functor

$$S: \text{Ord}^{\text{op}} \times \text{Ord}^{\text{op}} \to \text{SET},$$

i.e., an object in $\text{Set}^{\text{Ord}^{\text{op}} \times \text{Ord}^{\text{op}}}$.

Recall the geometric realization functor $|\ |: \text{SET}^{\text{Ord}^{\text{op}}} \to \text{TOP}$. This induces

$$|\ |\text{Ord}^{\text{op}}: (\text{SET}^{\text{Ord}^{\text{op}}})^{\text{Ord}^{\text{op}}} \to \text{TOP}^{\text{Ord}^{\text{op}}}.$$

Thus, since $\text{SET}^{\text{Ord}^{\text{op}}} \times \text{Ord}^{\text{op}} \simeq (\text{SET}^{\text{Ord}^{\text{op}}})^{\text{Ord}^{\text{op}}}$ as categories, a bisimplicial set gives rise to two simplicial spaces. They are $q \to |p \to S(p, q)|$ and $p \to |q \to S(p, q)|$ which will be denoted by S_1 and S_2 respectively. In addition, the diagonal functor Δ: $\text{Ord}^{\text{op}} \to \text{Ord}^{\text{op}} \times \text{Ord}^{\text{op}}$ given by $\Delta(n) = (n, n)$ on objects and $\Delta(\delta) = (\delta, \delta)$ on arrows induces a functor $\text{SET}^{\Delta}: \text{SET}^{\text{Ord}^{\text{op}} \times \text{Ord}^{\text{op}}} \to \text{SET}^{\text{Ord}^{\text{op}}}$. Then denote by $\Delta(S)$ the simplicial set $\text{SET}^{\Delta}(S)$.

Now there is a geometric realization functor $|\ |: \text{TOP}^{\text{Ord}} \to \text{TOP}$ defined in a manner analogous to the realization functor for simplicial sets. We then have the following result due to Tornehave.

Proposition 1.1.3 (Tornehave [12]). *$|S_1|$, $|S_2|$ and $|\Delta(S)|$ are homotopy equivalent.*

Proof of Theorem 1.1.1. Consider the category given by $DF \# Y$ whose objects are diagrams of the form

$$y \xrightarrow{b} F(x)$$

and whose arrows are given by commutative diagrams of the following form:

$$\begin{tikzcd} y_1 \arrow[r, "b_1"] & F(x_1) \arrow[d, "a"] \\ y_2 \arrow[u, "b"] \arrow[r, "b_2"] & F(x_2) \end{tikzcd}$$

In [6] it is shown that $|DF \# Y|$ is homotopy equivalent to $|X|$.

Next, consider the bisimplicial set $S(F)\colon \mathrm{Ord}^{\mathrm{op}} \times \mathrm{Ord}^{\mathrm{op}} \to \mathrm{SET}$ given by

$$S(F)\,(n, m) = (y_0[b_1, \ldots, b_n], x_0[a_1, \ldots, a_m]),$$

$$F(a_1\colon x_0 \to x_1) = b_n\colon y_{n-1} \to y_n.$$

Then it is clear that $|\Delta(S(F))| \sim_h |DF \# Y|$.

Now, $S(F)_2(n) = \coprod |DF(y)|_{y[a_1, \ldots, a_n]}$ the disjoint union ranges over n paths that terminate at y. But now, by a theorem of Segal (see [8]), there is a spectral sequence whose $E^2_{p,q}$ term is $K_p(H_q(S(F)_2))$ and whose termination is $H_{p+q}(|S(F)_2|)$ where $K_p(H_q(S(F)_2))$ denotes the homology of the simplicial abelian group $H_q(S(F)_2)$. But this is clearly the same as $H_p(Y^{\mathrm{op}}, H_q DF)$. □

Corollary 1.1.4. *Let $f\colon y^{\mathrm{op}} \to \mathrm{CAT}$ be given. Then we have a spectral sequence whose $E^2_{p,q}$ term is $H_p(Y^{\mathrm{op}}, H_q f)$ and whose termination is $H_{p+q}(|f * Y|)$. Here $H_q f\colon Y^{\mathrm{op}} \to AB$ is given by $H_q f(y) = H_q(|f(y)|, Z)$ on objects if $a\colon y_1 \to y_2$ is an arrow, then*

$$H_q f(a) = H_q(|a|, Z)\colon H_q(|f(y_2)|, Z) \to H_q(|f(y_1)|, Z).$$

Proof. This follows from Theorem 1.1.1 and Proposition 0.3.10. □

Corollary 1.1.5. *Given two monoids M and N, we have a spectral sequence whose $E^2_{p,q}$ term is $H_p(N^{\mathrm{op}}, H_q(\mathrm{Map}(N, M), Z))$ and whose termination is $H_{p+q}(M \circ N, Z)$.*

Proof. This follows from Corollary 1.1.4 and the fact that $M \circ N = f * N$ where $f\colon N \to \mathrm{End}(\mathrm{Map}(N, M))$. □

Remark. There are obvious dual theorems for cohomology spectral sequences.

1.2. Euler characteristics

Definition 1.2.1. Suppose $\{X_*, \partial_*\}$ is a finite chain complex. Then the *Euler characteristic of* $\{X_*, \partial_*\}$ is defined by $\chi(X) = \sum_{1 \geq 0}(-1)^i \mathrm{rank}(X_i)$.

Proposition 1.2.2. *Let $\{X_*, \partial_*\}$ be a finite chain complex. Then $\chi(X) = \sum_{1 \geq 0}(-1)^i \mathrm{rank}(H_i(X))$, where $H_i(X)$ denotes the ith homology group of the complex.*

Proof. Well known, classical, and can be found, for instance, in [10]. □

Now, suppose a given chain complex $\{X_*, \partial_*\}$ is chain homotopic to a finite chain complex $\{Y_*, \partial_*\}$. Then we define *the Euler characteristic of* $\{X_*, \partial_*\}$ to be the Euler characteristic of $\{Y_*, \partial_*\}$. Since chain homotopy is an equivalence relation on the set of chain complexes, and a chain homotopy induces an isomorphism on homology, we see by Proposition 1.2.2 that this definition is independent of the finite representative $\{Y_*, \partial_*\}$.

In the case of a topological space S, the Euler characteristic of S, $\chi(S)$, if it exists, is equal to the Euler characteristic of the simplicial chain complex giving the homology of S. At least the set of spaces that are homotopy equivalent to a finite CW-complex will have Euler characteristics defined for them.

Similarly, we define the Euler characteristic of a category C, $\chi(C)$, if it exists, to be the Euler characteristic of $|C|$. For monoids, the Euler characteristic will be defined for the set of monoids satisfying condition D'_n for $n > 3$.

In what follows, all categories considered will have finitely many objects and arrows.

Now, let $f: X \to Y$ be a functor. For each $y \in \mathrm{Obj}(Y)$, denote by $\#_n(y)$ the number of paths of length n that terminate at y. In other words, $\#_n(y)$ is the cardinality of

$$y_n \xrightarrow{a_n} y_{n-1} \xrightarrow{a_{n-1}} \cdots\ y_1 \xrightarrow{a_1} y.$$

Definition 1.2.3. We say that a category C has *finite homological dimension* if there exists $n \in Z^+$ such that for all $F: C \to AB$, $H_m(C, F) = 0$ for all $m > n$.

Proposition 1.2.4. *Suppose $F: X \to Y$ is a functor from a category into a poset with $H_q(|DF(y)|, Z)$ finitely generated for all p and $H_q(|DF(y)|, Z) = 0$ if q is large enough. Then we have the following formula*:

$$\chi(X) = \sum_{i+j=n} (-1)^n \sum_{y \in \mathrm{Obj}(Y)} \#_i(y)\beta_j(|DF(y)|),$$

where $\beta_j(|DF(y)|) = \mathrm{rank}(H_j(|DF(y)|, Z))$.

Proof. By hypothesis, we see that $\{E_n^2, \partial_*\}$ where $E_n^2 = \bigoplus_{i+j=n} E_{i,j}^2$ is a finite complex, i.e., $E_n^2 = 0$ for n large enough. Thus, by Proposition 1.3.2, we have $\chi(|X|) = \sum_{i+j=n \geq 0}(-1)^n \mathrm{rk}(E_{i,j}^2)$. Now, for $i > 0$,

$$\mathrm{rk}(E_{i,j}^2) = \mathrm{rk}(H_i(Y^{\mathrm{op}}, H_j DF)) = \mathrm{rk}(\mathrm{Ker}\,\partial_{i,j}^2) - \mathrm{rk}(\mathrm{Im}\,\partial_{i+1,j}^2)$$

and

$$\mathrm{rk}(E_{0,j}^2) = \sum_{y \in \mathrm{Obj}(y)} \mathrm{rk}(H_j(|DF(y)|, Z) - \mathrm{rk}(\mathrm{Im}\,\partial_{i+1,j}^2),$$

where

$$\partial_{i,j}^2 : \bigoplus_{y[a_1, \ldots, a_i]} H_j(|DF(y)|), Z) \to \bigoplus_{y[a_1, \ldots, a_{i-1}]} H_j(|DF(y)|), Z)$$

is the boundary operator. But then for $i > 0$,

$$\mathrm{rk}(\mathrm{Ker}\,\partial^2_{i,j}) = \mathrm{rk}\left(\bigoplus_{y[a_1,\ldots,a_i]} H_j(|DF(y)|), Z)\right) - \mathrm{rk}(\mathrm{Im}\,\partial^2_{i,j})$$

$$= \sum_{y \in \mathrm{Obj}(Y)} \#_i(y)\beta_j(|DF(y)|) - \mathrm{rk}(\mathrm{Im}\,\partial^2_{i,j}).$$

Thus,

$$\mathrm{rk}(E^2_{i,j}) = \sum_{y \in \mathrm{Obj}(Y)} \#_i(y)\beta_j(|DF(y)|) - \mathrm{rk}(\mathrm{Im}\,\partial^2_{i,j}) - \mathrm{rk}(\mathrm{Im}\,\partial^2_{i+1,j})$$

for $i > 0$.

Hence by the fact that we have a telescoping summation, we obtain

$$\chi(X) = \sum_{i+j=n} (-1)^n \sum_{y \in \mathrm{Obj}(Y)} \#_i(y)\beta_j(|DF(y)|). \qquad \square$$

Example 1.2.5. Suppose M has finite homological dimension and X is a finite right M-set with $|X//M|$ homotopy equivalent to a finite CW-complex. Then consider the functor $\Omega_M : X//M \to \Omega_M(X)$ as in Section 0. Then by Proposition 1.2.4,

$$\chi(X//M) = \sum_{i+j=n} (-1)^{i+j} \sum_{O(x)} \#_i(O(x))\beta_j(|O(x)//M|),$$

where $\#_i(O(x))$ is the number of nested sequences of the form

$$O(x) \subseteq O(x_1) \subseteq \cdots \subseteq O(x_i)$$

and $\beta_j(|O(x)//M|) = \mathrm{rank}(H_j(M, ZO(z)))$.

Example 1.2.6. Suppose $f : M \to N$ is a monoid homomorphism and $R_f : R_M \to R_N$ is the induced map between the respective R-order posets. Then we have

$$\chi(R_M) = \sum_{i+j=n} (-1)^{i+1} \sum_{Nn} \#_i(Nn)\beta_j(|R\downarrow(Nn)|, Z),$$

where $\#_i(Nn)$ is the number of nested sequences of the form

$$Nn \subseteq Nn_1 \subseteq Nn_2 \subseteq \cdots \subseteq |Nn_i$$

and $R_f\downarrow(Nn)$ denotes the subposet of R_M given by the $\{Mm' | Mm' \subseteq Mm$ for all $M \in R_f^{-1}(Nn)\}$.

Corollary 1.2.7. *Suppose $F : X \to Y$ satisfies the conditions of Proposition 1.2.4 and that in addition the spaces $|DF(y)|$ all have the same homology and are homotopy equivalent to a finite CW-complex. Then $\chi(X) = \chi(Y)\chi(|DF(y)|)$ for any $y \in Y$.*

Proof. By Proposition 1.2.4 we have

$$\chi(X) = \sum_{i+j=n} (-1)^{i+j} \sum_{y} \#_i(y)\beta_j(|DF(y)|).$$

But since $\beta_j(|DF(y)|)$ is independent of y, it factors out of the inner summation, and we get

$$\chi(X) = \sum_{i+j=n} (-1)^{i+j} \#_i \beta_j(|DF(y)|) = \left(\sum_{i \geq 0} (-1)^i \#_i\right)\left(\sum_{j \geq 0} (-1)^j \beta_j(|DF(y)|)\right)$$

for any fixed y and where $\#_i$ is the number of i-chains in Y. But since Y is a finite poset, we see that

$$\xi(Y) = \sum_{i \geq 0} (-1)^i \#_i. \quad \square$$

1.3. Acyclicity conditions for Z^0 and the R-order poset

For a given monoid M denote by $M_{<1}$ the ideal of elements J below one. Then M acts on Z on the right by $na = 0$ if $a \in M_<$ and $na = n$ otherwise. To distinguish this ZM-module from the usual "trivial" action of ZM on Z, we denote this module by Z^0.

Note. In the following, all monoids M will have the property that $M - M_< = 1$.

In the next section, we will be interested in cases where Z^0 is an acyclic module. One way of recovering the homology of Z^0 from ZM itself is by considering the following short exact sequence of right ZM-modules:

$$0 \longrightarrow ZM_< \longrightarrow ZM \longrightarrow ZM/ZM_< \longrightarrow 0.$$

It is clear that $ZM/ZM \simeq Z^0$ as modules. Also, by the Key Lemma, $H_i(M, ZM) = H_i(|M//M|, Z)$ for $i > 0$. Since $1 \in M$ is an initial object for $M//M$, we see that $|M//M|$ is contractible and hence is acyclic. Therefore, ZM is acyclic and by the long exact homology sequence, we obtain the following proposition.

Proposition 1.3.1. $H_i(M, Z^0) \simeq H_{i-1}(M, ZM_<) \simeq H_{i-1}(|M_<//M|, Z)$.

Denote by R_M the R-order poset for $M_<$. Then we have the following theorem.

Theorem 1.3.2. *Z^0 is acyclic if $|R_{M_<}|$ and ZmM are acyclic for all principal right ideals mM.*

Proof. We have a projection $\Omega: M_<//M \to R_M$ as described in Section 0. It is easy to see that if $[m]$ denotes an R-class represented by m, then $D\Omega([m]) = mM//M$.

Furthermore, if $[m_2] > [m_1]$, then $D\Omega([m_2] > [m_1]): m_1 M//M \to m_2 M//M$ is just the inclusion on objects.

Now, by Theorem 1.1.1, there is a spectral sequence whose $E^2_{p,q}$ term is $H_p(R^{\mathrm{op}}_M, H_q D\Omega)$ are whose termination is $H_{p+q}(|M_<//M|, Z)$. Therefore, $E^2_{p,q} = 0$ if $q > 0$ by the acyclicity assumption on ZmM and $E^2_{p,q} = H_q((|R^{\mathrm{op}}_M|), Z) = H_p(|R_M|, Z) = 0$ by the acyclicity assumption on $|R_M|$. Hence, $H_n(|M_<//M|, Z) = 0$ and by the above proposition, Z^0 is acyclic. □

Also, suppose $f: M \twoheadrightarrow N$ is a homology equivalence in the sense that if E is any right ZN-module, the induced map $H_*(f, E): H_*(M, E) \to H_*(N, E)$ is an isomorphism. Then we have the following proposition.

Proposition 1.3.3. *Suppose that ZmM is ZM acyclic for all $m \in M$ and ZnN is Zn acyclic for all $n \in N$. Then the induced map*

$$H_*((|R(f)|), Z): H_*(|R_M|), Z) \longrightarrow H_*(|R_N|), Z)$$

is an isomorphism.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} M_<//M \arrow[r] \arrow[d] & N_<//N \arrow[d] \\ R_M \arrow[r, "R_f"] & R_N \end{tikzcd}$$

Then since f is a homology equivalence, we see by Proposition 1.3.1 that when we apply the homology functor to the top arrow in the above diagram, we get an isomorphism.

Associated with the vertical arrows are spectral sequences whose $E^2_{p,q}$ terms for $q > 0$ by our acyclicity assumptions. Thus when we apply the homology functor to the vertical arrows we get isomorphisms. Thus, when we apply the homology functor to the bottom horizontal arrow, we get an isomorphism. □

2. (Co)homology equivalences and null surmorphisms

2.1. (co)homology equivalences and the Key Lemma

Recall that a monoid homomorphism $f: M \to N$ is said to be a homology (resp. cohomology) equivalence if for all right (resp. left) ZN modules E,

$H_*(f, E): H_*(M, E) \to H_*(N, E)$ is an isomorphism

(resp. $H^*(f, E): H^*(N, E) \to H^*(M, E)$ is an isomorphism).

By a corollary of the Mapping Theorem in homological algebra (see [3, p. 189], a surmorphism $f: M \twoheadrightarrow N$ is a (co)homology equivalence iff $H_i(M, ZN) = 0$ for all $i > 0$. (Here we view N as a *right* M-set via f.) Thus by the Key Lemma, we obtain the following proposition.

Proposition 2.1.1. *A surmorphism $f: M \twoheadrightarrow N$ is a (co)homology equivalence iff $|Df|$ is acyclic as a topological space.*

Proof. $H_i(M, ZN) = H_i(|N//M|), Z) = H_i(|Df|, Z)$ by the Key Lemma, so we are done by the above remarks. $\square$

Example 2.1.2. Consider the natural projection $P: M \circ N \twoheadrightarrow N$. Then by proposition 0.3.10, we see that $|DF| \sim_h |\mathrm{Map}(N, M)|$. Therefore P is a (co)homology equivalence iff $|M|$ is acyclic. For example, $P: U_1 \circ M \twoheadrightarrow M$ is a (co)homology equivalence for all M, by Example 0.3.3.

Example 2.1.3. Let $f: G \twoheadrightarrow H$ be a group surmorphism. Denote by K the kernel of f. Then by Proposition 0.3.13, we know that $|Df|$ is a $K(K, 1)$ space and therefore f is a (co)homology equivalence iff $|K|$ is acyclic. In particular, N is perfect.

Now let $f: M \twoheadrightarrow N$ be a fixed surmorphism of monoids. The adjective *f-singular*, when applied to any subset $U \subseteq M$ possibly with structure, will mean that f restricted to $M - U$ is injective. For instance, we may speak of f-singular ideals, J-classes, H-classes, etc.

Let $I \subseteq M$ be an f-singular ideal of M. Also, let C denote the full subcategory of $N//M$ generated by all arrows between the set of objects $K^{\bullet}$. Then note that we have the following diagram of functors:

$$K^{\bullet}//I^{\bullet} \xrightarrow{T} C \xrightarrow{S} N//M,$$

where $K = f(I)$. Then we have the following important theorem.

Theorem 2.1.4. *If f restricted to $I^{\bullet}$ is a (co)homology equivalence, then f is a (co)homology equivalence.*

Proof. By Proposition 2.1.1, we see that if the restriction of f to $I^{\bullet}$ is a (co)homology equivalence, then $|K^{\bullet}//I^{\bullet}|$ is acyclic.

We claim that this implies that $|C|$ is acyclic. By Theorem 1.1.1, we have a spectral sequence associated with $T: K^{\bullet}//I^{\bullet} \to C$ whose $E^2_{p,q}$ term is $H_p(C^{\mathrm{op}}, H_q DT)$ and whose termination is $H_{p+q}(|K^{\bullet}//I^{\bullet}|, Z)$. To prove the first claim, we show that $E^2_{p,q} = H_p(K^{\bullet}//I^{\bullet}, H_q DT) = 0$ for $q > 0$. When $k = 1 \in \mathrm{Obj}(C)$, note that $DT(1) \simeq I^{\bullet}//I^{\bullet}$, and so $H_q DT(1) = 0$ for all $q > 0$. For every $1 \neq k \in \mathrm{Obj}(C) = K^{\bullet}$, we

$$\begin{tikzcd}
& Z_qDT \arrow[rr, "P_1"] \arrow[dd, "Q_1"] & & Z_qDT/\mathrm{Im}(\partial_{q-1}) \arrow[dd, "Q_2"] \\
\mathrm{Ker}\,\partial_q(|DT|) \arrow[ur, "i_1"] \arrow[rr, "P_2"] \arrow[dd] & & H_qDT \arrow[ur, "j_1"] \arrow[dd, "Q_4"] & \\
& Z_qDT/A_q(DT(k) \arrow[rr, "P_3"] & & Z_qDT/(\mathrm{Im}(\partial_{q+1})+A_q(DT(k))) \\
\mathrm{Ker}\,\partial_q(|DT|)/A_q(DT(k) \arrow[ur, "i_2"] \arrow[rr, "P_4"] & & \mathrm{rel}(H_qDT) \arrow[ur, "j_2"] &
\end{tikzcd}$$

Fig. 1

have a projection functor $P(k):DT(k) \to M//I^\bullet$ given by $P(k)$ $(m, kf(m)) = m$ on objects and $P(k)\,(k':(m_1, kf(m_1)) \to (m_2, kf(m_2)) = k':m_1 \to m_2$ on arrows. Since I is an ideal, we see that $P(k)$ is a categorical isomorphism. Thus we can represent an element $\sigma \in H_p(C, H_qDT)$ in the following way:

$$\left(\sum_i (-1)^{\beta i,j} k^{i,j}[1_1^{i,j}, \dots, 1_q^{1,j}],\ \sum_i (-1)^{\alpha i} k^i [m_1^i, \dots, m_p^i]\right),$$

where $\sum_j(-1)^{\beta i,j} m^{i,j}[1^{i,j}, \dots, 1^{i,j}]$ represents an element in

$H_q(|DT(k_p)|, Z) = H_q(|M//I^\bullet|, Z)$ and $m_p: k_{p-1} \to k_p$.

Furthermore, $H_qDT: C \to AB$ is given in the following way: suppose $m: k_1 \to k_2$ is an arrow in C and $\sum_i(-1)^{\alpha i} m^i[1_1^i, \dots, 1_q^i]$ represents an element of $H_qDT(k_2)$. Then

$$H_qDT(m)\left(\sum_i (-1)^{\alpha i} m^i[1_1^i, \dots, 1_q^i]\right) = \sum_i (-1)^{\alpha i} m m^i [1_1^i, \dots, 1_q^i]$$

and it is easy to see that this is independent of choice of representatives.

Since $|I^\bullet//I^\bullet|$ is acyclic, we have by the long exact homology sequence for $(|I^\bullet//I^\bullet|Z, |M//I^\bullet|)$ that $H_p(|M//I^\bullet|, Z) \simeq H_p(|M//I^\bullet|, |I^\bullet//I^\bullet|, Z)$. Now, consider the diagram in $AB^{C^{\mathrm{op}}}$ shown in Fig. 1.

Here: (1) $Z_qDT: C^{\mathrm{op}} \to AB$ is given by $Z_qDT(k) = ZNDT(k)(q)$, the free abelian group on elements of the form $m[1_1, 1_2, \dots, 1_q]$ with $1_i \in I$. (2) $A_q(DT(k)$ denotes the subgroup of $ZNDT(k)(q)$ generated by elements of the form $1[1_1, 1_2, \dots, 1_q]$ with $1 \in I$. (3) $\mathrm{Im}(\partial_{q+1})$ is the image of $\partial_{q+1}: Z_{q+1}DT(k) \to Z_qDT(k)$. (4) $\mathrm{Ker}(\partial_q)$ is the kernel of $\partial_q: Z_qDT(k) \to Z_{q-1}DT(k)$. (5) $\mathrm{rel}(H_qDT) = H_q(|M//I^{\boldsymbol{\cdot}}|, |I^{\boldsymbol{\cdot}}//I^{\boldsymbol{\cdot}}|, Z)$. (6) The arrows in C act on all of the above functors by left multiplication just as in the above description for H_qDT. Note in particular that we used the fact that I is an ideal for the fact that the lower commutative square is really in $AB^{C^{\mathrm{op}}}$.

Now, by tensoring the above diagram over C^{op} with the resolution given by the GKNT adjunction as in Section 0, we obtain a corresponding commutative diagram of chain complexes. For simplicity denote by $\{G_*, \partial_*\}$ the resolution given by GKNT. Then consider $z \in Z_qDT \otimes_{C^{\mathrm{op}}} G_{p+1}$ given by

$$z = \left(\sum_j (-1)^{\beta i,j} [1_1^{i,j}, \dots, 1_q^{i,j}], \ \sum_i (-1)^{\alpha i} k^i [m_1^i, \dots, m_p^i, m_{p+1}^{i,j}] \right).$$

It is clear that

$$j_2 \otimes_{C^{\mathrm{op}}} G_p(Q_4 \otimes_{C^{\mathrm{op}}} G_p(\sigma)) = P_3 \otimes_{C^{\mathrm{op}}} G_p(Q_1 \otimes_{C^{\mathrm{op}}} G_p \partial_{p+1}(z)).$$

Thus since Q_4 is an isomorphism, we see that $\sigma = 0$ in homology.

Our next step is to show that $S: C \to N//M$ induces an epimorphism on homology. Suppose $n[m_1, m_2, \dots, m_p] \in Z_qN//M$ with $n > 1$. Then choose $m \in f^{-1}(n)$. We see that

$$\partial_{p+1}(1[m, m_1, m_2, \dots, m_p]) = n[m_1, m_2, \dots, m_p] + z,$$

where $z \in A_q(N//M)$. Thus if $\sigma \in H_p(|N//M|, Z)$, then we can find a representative for it of the following form:

$$x = \sum_i (-1)^{\alpha i} 1[m_1^i, \dots, m_p^i].$$

Now since $\partial_p(x) = 0$, we see that for each i, there is a j so that $f(m^i)[m_1^i, \dots, m_p^i] = f(m^j)[m_1^j, \dots, m_p^j]$. In particular, we may assume that $m^i \neq m^j$. But this means that $m^i, m^j \in I$, since I was assumed to be f-singular. Since I is an ideal, we see that $x \in Z_q(C)$, which is what we wanted to prove. Thus we see that $|N//M|$ is acyclic and by Theorem 2.1.4, f is a (co)homology equivalence. □

One sees by examining the proof that actually more was proved, namely Proposition 2.1.5.

Proposition 2.1.5. *Suppose $f: M \twoheadrightarrow N$ is a surmorphism and I is an f-singular ideal. Then the functor $U: K^{\boldsymbol{\cdot}}//I^{\boldsymbol{\cdot}} \to N//M$ induces an epimorphism on homology in Z coefficients.*

Corollary 2.1.6. *If f is not a homology equivalence, then the restriction of f to $I^{\boldsymbol{\cdot}}$ is not homology equivalence.*

2.2. *Null surmorphisms are (co)homology equivalences*

A surmorphism $f: M \twoheadrightarrow N$ is said to be a *maximal proper surmorphism* (or MPS for short) if there do not exist nonidentity surmorphisms $g: M \twoheadrightarrow L$ and $h: L \twoheadrightarrow N$ such that $f = hg$. In [7] it is shown that for finite monoids, every surmorphism factors into MPSs and in fact there are precisely ten types of MPSs. Intuitively, the various types reflect the fact that different f-singular A-classes arise, where A is one of the Green's relations, and these f-singular classes may be regular or null.

It is known from [7] that every MPS has precisely one or two f-singular J-classes. For a given MPS, $f: M \twoheadrightarrow N$, we will denote by I_f the ideal generated by the f-singular J-classes. We will also denote by J_f a maximal f-singular J-class.

Definition 2.2.1. An MPS $f: M \twoheadrightarrow N$ is said to be *null* if $f^{-1}(f(J_f))$ is a union of null J-classes.

Remark. Using the notation in [7], it is easy to see that null MPSs are precisely those of type I(n), II(n), II(n), III($n > n$), and IV.

Definition 2.2.2. Any surmorphism $f: M \twoheadrightarrow N$ that factors into null MPSs is called a *null* surmorphism.

Theorem 2.2.3. *Let $f: M \twoheadrightarrow N$ be a null MPS. Then f is a (co)homology equivalence.*

Corollary 2.2.4. *A null surmorphism is a (co)homology equivalence.*

Proof of Theorem 2.2.3. Let $K_f = f(I_f)$. Denote by $g: I_f^{\bullet} \twoheadrightarrow K_f^{\bullet}$ the restriction of f to $I_f^{\bullet}$. Then by Theorem 2.1.4. it is enough to show that g is a (co)homology equivalence. To this end, we actually prove something stronger, the following chain.

Claim. *$|Dg| = |K_f^{\bullet}//I_f^{\bullet}|$ is contractible.*

Define $L_f = f(J_f)$ and $W_f = f^{-1}(L_f)$. Define choice functions as follows.

Case 1: f is of type III($n > n$). Then for each $n \in L_f$, *choose* $m_n \in f^{-1}(n) - J_f$.

Case 2: f is not of type III($n > n$). Then for each $n \in L_f$, *choose* $m_n \in f^{-1}(n)$.
In both these cases, denote by c this choice function.
Let B Arrows(Dg) be given by the following set:

$$B = \{m: 1 \longrightarrow g(m) | m \in W_f - c(L_f)\}.$$

We claim G_g given by $\text{Obj}(G_g) = \text{Obj}(Dg)$ and $\text{Arr}(G_g) = \text{Arr}(Dg) - B$ is a subcategory of Dg. In fact, the only thing that needs verification is that if $c(n): 1 \to n$ and $m: n \to ng(m)$ are given, then $c(n)m: 1 \to ng(m)$ is the *only* arrow from 1 to $ng(m)$. But this follows from the fact that W_f is a union of f-singular null J-classes.

Now note that 1 is an initial object in G_g, so by Corollary 0.3.7 we see that $|G_g|$ is contractible. We next consider the (relative CW) NDR pair ($|Dg|, |G_g|$). By the long exact sequence of homotopy groups associated with this pair we thus see that

$\pi_i(|Dg| \simeq \pi_i(|Dg|, |G_g|)$ for all i. In particular, $\pi_1(|Dg|) \simeq \pi_1(|Dg|, |G_g|)$. But $\pi_1(|Dg|, |G_g|)$ is generated by 1-simplices indexed by arrows of the form $m: 1 \to f(m) \in B$. However, these 1-simplices are bound by 2-simplices indexed by diagrams of the following form:

$$1 \xrightarrow{m} f(m) \xrightarrow{m'} f(mm') \quad \text{such that } 1 \xrightarrow{m} f(m) \in B.$$

This follows from the fact that nullity of W_f implies that $mm': 1 \to f(mm')$ is an arrow in G_g. Thus we see that

$$\pi_1(|Dg|) = \pi_1(|Dg|, |G_g|) = 0.$$

By the Hurewicz Theorem, $|Dg|$ will be contractible if it is acyclic (which is what we originally wanted to prove). Now, by the Key Lemma, $H_i(|Dg|, Z) = H_i(I^{\cdot}_f, ZK^{\cdot}_f)$ for $i > 0$. Let $A_f = I^{\cdot}_f - W^{\cdot}_f = K^{\cdot}_f - L^{\cdot}_f$ and consider the monoids $W^{\cdot}_f \cup \{x\}$ and $L^{\cdot}_f \cup \{x\}$ where $mn = x$ if $m, n \neq 1$. We have following commutative diagram:

$$\begin{array}{ccc} I^{\cdot}_f & \longrightarrow & K^{\cdot}_f \\ \downarrow & & \downarrow \\ I^{\cdot}_f \cup \{x\} & \longrightarrow & K^{\cdot}_f \cup \{x\} \end{array}$$

This induces a corresponding diagram of short exact sequences of right $ZI^{\cdot}_f$ modules (see Fig. 2). We claim that by the four indicated isomorphisms, it will be enough to show that Zf_x induces an isomorphism on homology. For suppose this were the case. Then considering the lower part of the diagram, we obtain an associated morphism of long exact homology sequences as in Fig. 3. Then if $H_p(Zf_x)$ is an isomorphism, so is $H_p(g_1)$. But then this implies that $H_p(g_2)$ is an isomorphism and thus so is $H_p(Zg)$. Since $H_p(I^{\cdot}_f, ZI^{\cdot}_f) = 0$, we would be done by the Key Lemma.

Now consider the functor

$$f_x: W^{\cdot}_f \cup \{x\}//I^{\cdot}_f \to L^{\cdot}_f \cup \{x\}//I^{\cdot}_f.$$

Then notice that, by the nullity assumption, $Df_x(m) \simeq I^{\cdot}_f//I^{\cdot}_f$ if $m \neq 1$. This implies that the $E^2_{p,q}$ term of the spectral sequence associated with this functor is 0 if $q > 0$ and $p > 0$. We also have

$$E^2_{p,0} = E_{p,0} = H_p(|L^{\cdot}_f \cup \{x\}//I^{\cdot}_f|, Z) \quad \text{and} \quad E^2_{0,q} = H_q(|Df_x(l)|, Z).$$

Note that any $\sigma \in H_p(|L^{\cdot}_f \cup \{x\}//I^{\cdot}_f|, Z)$ can be written in the form

$$\sigma = \sum_i (-1)^{\alpha i} 1[m^i_1, \ldots, m^i_p]. \tag{$*$}$$

Since $\partial_p(\sigma) = 0$ we see that for every $i > 0$, there exists j so that $f(m^i)[m^i_1, \ldots, m^i_p] = f(m^j)[m^j_1, \ldots, m^j_p]$. Thus we can write $(*)$ in the following form:

$$\sum_i (-1)^{\alpha i}(1[m^i_1, m^i_2, \ldots, m^i_p] - 1[m'^i_1, m^i_2, \ldots, m^i_p])$$

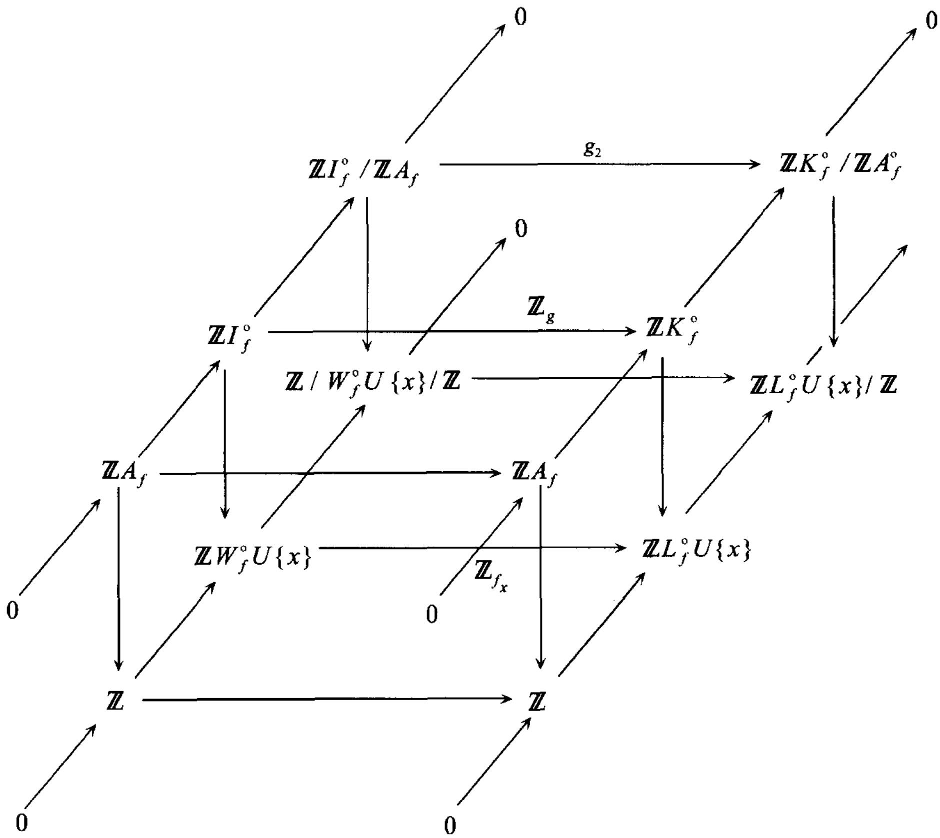

Fig. 2

with $f(m^i) = f(m'^i)$. Note that $E^2_{0,q-1} = E^p_{0,q-1}$ if $p < q - 1$.
Furthermore, $d^q_{q,0}: E^{q-1}_{q,0} \to E^{q-1}_{0,q-1}$ is given by

$$d_{q,0}\left(\sum(-1)^{\alpha i}(1[m_1^i, m_2^i, \ldots, m_q^i] - 1[m_1'^i, m_2^i, \ldots, m_q^i])\right)$$

$$= \sum_i (-1)^{\alpha i}(m_1^i, m_1'^i)[m_2^i, \ldots, m_q^i],$$

where $(m_1^i, m_1'^i) \in \mathrm{Obj}(DT(1))$. But this map is clearly onto, and we see that $E^q_{0,q-1} = 0$. Thus, by induction, in the limit we have $E^2_{p,0} = H_p(|L^\cdot_f \cup \{x\}//I^\cdot_f|, Z) = H_p(|W^\cdot_f \cup \{x\}//I^\cdot_f|, Z)$. We are then done by the Key Lemma since this implies that Zf_x induces an isomorphism on homology. $\square$

$$
\begin{tikzcd}
\cdots \arrow[r] & H_{p+1}(I_f^\circ, \mathbb{Z}) \arrow[r] \arrow[d] & H_{p+1}(I_f^\circ, \mathbb{Z}(W_f^\circ U \{x\})) \arrow[r] \arrow[d, "H_{p+1}(\mathbb{Z}f_x)"] & H_{p+1}(I_f^\circ, \mathbb{Z}(W_f^\circ U \{x\}/\mathbb{Z}) \arrow[r] \arrow[d, "H_p(g_1)"] & \cdots \\
\cdots \arrow[r] & H_{p-1}(I_f^\circ, \mathbb{Z}) \arrow[r] & H_{p+1}(I_f^\circ, \mathbb{Z}(L_f^\circ U \{x\})) \arrow[r] & H_{p+1}(I_f^\circ, \mathbb{Z}(L_f^\circ U \{x\}/\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}
$$

Fig. 3

2.3. *The type* III $(N > R)$ *case*

Proposition 2.3.1. *Suppose* $f: M \twoheadrightarrow N$ *is a type* III$(N > R)$ *MPS. If* Z^0 *is an acyclic* $ZI_f^\bullet$ *module, then* f *is a (co)homology equivalence.*

Proof. By Theorem 2.1.4, it is enough to show that f restricted to $I_f^\bullet$ is a (co)homology equivalence. To this end, we show that $ZK_f^\bullet$ is $ZI_f^\bullet$ acyclic where $K_f^\bullet = f(I_f^\bullet)$. Note that K_f can be identified with the deal strictly J below J_f, which is the maximal f-singular J-class.

Consider the following short exact sequence of $ZI_f^\bullet$ modules:

$$0 \longrightarrow ZK_f \longrightarrow ZI_f \longrightarrow ZI_f/ZK_f \longrightarrow 0.$$

Now, by nullity, it is clear that ZI_f/ZK_f can be identified with the k-fold direct sum of Z^0, $Z^0 + \cdots + Z^0$, where k is the cardinality of J_f. We first claim that any direct sum of Z^0's is acyclic. Consider the short exact sequence of $ZI_f^\bullet$ modules:

$$0 \longrightarrow Z^0 \longrightarrow Z^0 + Z^0 \longrightarrow Z^0 \longrightarrow 0.$$

By the long exact sequence associated with this, we see that $Z^0 + Z^0$ is acyclic. Then suppose the p-fold direct sum $Z^0 + \cdots + Z^0$ is acyclic. Then by the exact sequence

$$0 \longrightarrow (Z^0 + \cdots + Z^0) \longrightarrow (Z^0 + \cdots + Z^0) + Z^0 \longrightarrow Z^0 \longrightarrow 0.$$

we see that $(Z^0 + \cdots + Z^0) + Z^0$ is acyclic. This completes the induction. Thus we see that $ZK_f \to ZI_f$ induces an isomorphism on homology by the long exact sequence.

Now consider the following morphism of short exact sequences:

$$
\begin{tikzcd}
0 \arrow[r] & ZI_f \arrow[r] & ZI_f^\bullet \arrow[r] & Z^0 \arrow[r] & 0 \\
0 \arrow[r] & ZK_f \arrow[r] \arrow[u] & ZK_f^\bullet \arrow[r] \arrow[u] & Z^0 \arrow[r] \arrow[u] & 0
\end{tikzcd}
$$

By the induced morphism of long exact homology sequences associated with this diagram, we see that $ZK_f^\bullet$ is acyclic. □

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